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MM Double Exponential Distribution

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Abstract: The exponential distribution is one of the most significant and widely used distributions in statistical practices. In this paper, we introduce MM Double Exponential Distribution (MMDED) and find some properties with application to real life data. Maximum Likelihood Estimation (MLE) has been used to estimate the parameters of MMDED. Finally, we provide results of entropies and compareMMDED with other distributions for best fitted.

Keywords: Exponential distribution, moments, estimation and entropy.

1.1 INTRODUCTION

The Exponential distribution has a fundamental role in describing a large class of phenomena, mostly in the region of reliability theory. This distribution is commonly used to model waiting times between occurrences of rare events, lifetimes of electrical or mechanical devices. It is also used to get approximate solutions to difficult distribution problems.

The notion of MM-Double Exponential Distribution (MMDED) is introduced byMubbasher andZahida(2016). In this paper a new exponential distribution is presented which is known as MM Double Exponential Distribution.

The MMDED is the distribution of differences betweentwo independent variates with identical exponential distribution. It is a continuous probability distribution. It is defined by the location and scale parameters. The main characteristic of this distribution is that it is unimodel and symmetric. The dispersion of the data around the mean is higher than that of a normal distribution.

Actually an area where the MMDED and related probability distributions can find most interesting and successful application is on modeling of financial data. This distribution is also used for modeling and signal processing, various biological process and economics. Examples of event that may be modeled by MMDED includes

a) Exotic optional and credit risk in financial engineering
b) Insurance claims
c) Structural changes in switching regime model.

MMDED is used for fitting data in economics and health sciences. MMDED is a popular topic in probability theory due to simplicity of its characteristic function and density. MMDED is used not its own merits but as a source for counter examples for other (mainly normal) distributions. It has been used for purposes to provide examples of curiosity, non-regularity and pathological behavior. In studies with probabilistic content, the distribution serves as a tool for limiting theorems and representations with the emphasis on analyzing its differences from the classical theory based on the sound foundations of normality.

1.1.1 MM-Double Exponential Distribution (MMDED)

The MM-double exponential distribution is defined as:

$$g(x;\lambda,c) = \frac{f(x)F(cx)}{\int_0^\infty f(x)F(cx)dx},\ x \geq 0,\ \lambda, c > 0 \quad (1.1)$$

where $f(x) = \lambda e^{-\lambda x}, \lambda > 0,\ x \geq 0$ is the pdf of exponential distribution and

$F(cx;\lambda) = \frac{-1}{c}(e^{-\lambda cx} - 1)$

Thus the pdf of MMDED is

$$g(x;\lambda,c) = \frac{1+c}{c}\left[\lambda e^{-\lambda x}\left(1 - e^{-\lambda cx}\right)\right],\ x \geq 0,\ \lambda, c > 0$$

where$\lambda$is shape parameter and c is scale parameter.

(1.2)

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1.2 GRAPHS OF PROBABILITY DENSITY FUNCTION

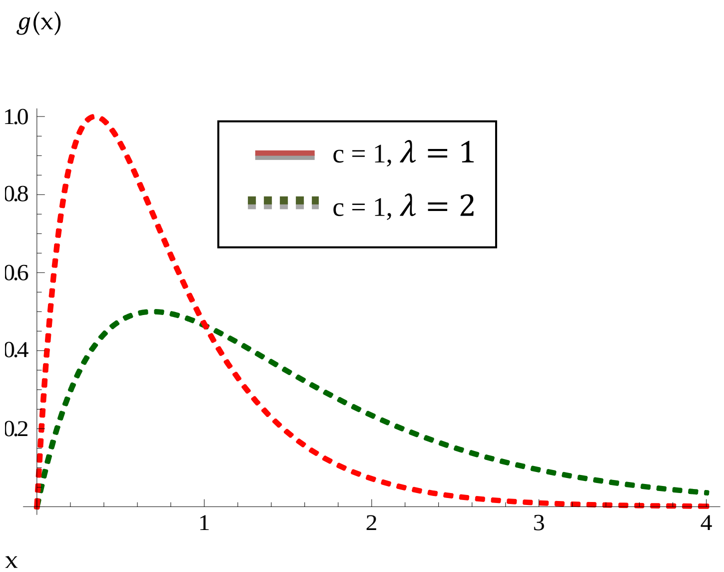

Fig. 1.2.1: The Probability Density Function of MMDED for the indicated values of c and λ

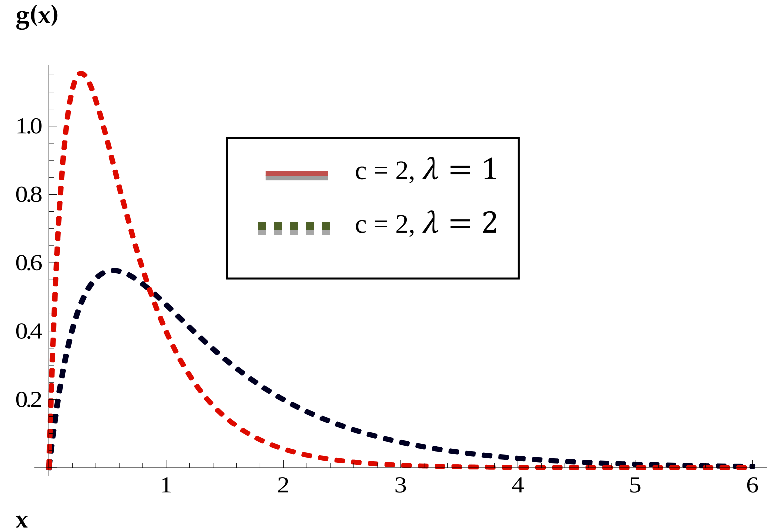

Fig. 1.2.2: The Probability Density Function of MMDED for the indicated values of c and λ

1.3 DISTRIBUTION FUNCTION OF MMDED

Distribution function of a density function is defined as:

$$G\,(x;\lambda,c) = \int_0^x g(t)dt$$

$$G\,(x) = \frac{\lambda(1+c)}{c}\int_0^x e^{-\lambda t}\,(1-e^{-\lambda ct})\,d\,t$$

After some simplification we have:

$$G(x) = 1 - \frac{1+c}{c\,e^{\lambda x}} + \frac{1}{ce^{\lambda x\,(1+c)}},\ x \geq 0,\ \lambda, c > 0 \qquad (1.3)$$

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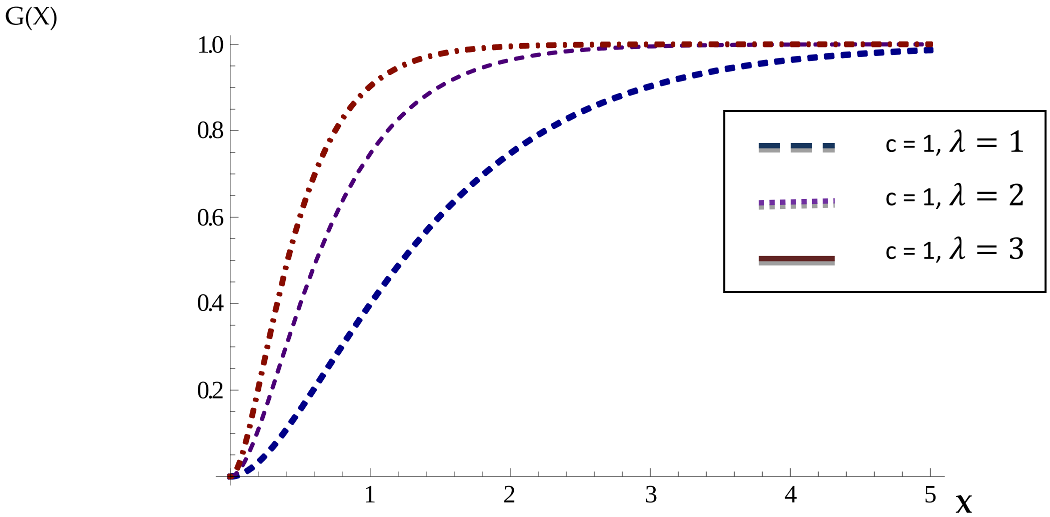

Fig. 1.3.1: **Distribution Function of MMDED for indicated values of c and λ**

1.4 SURVIVAL FUNCTION

The survival function is an important measure in a reliability studies, therefore by definition, the survival function for MMDED is:

$$\bar{G}(x) = 1 - G(x)$$

$$\bar{G}(x) = \frac{1+c}{c\, e^{\lambda x}} - \frac{1}{c e^{\lambda x(1+c)}}, \; x \geq 0, \lambda, c > 0 \quad (1.4)$$

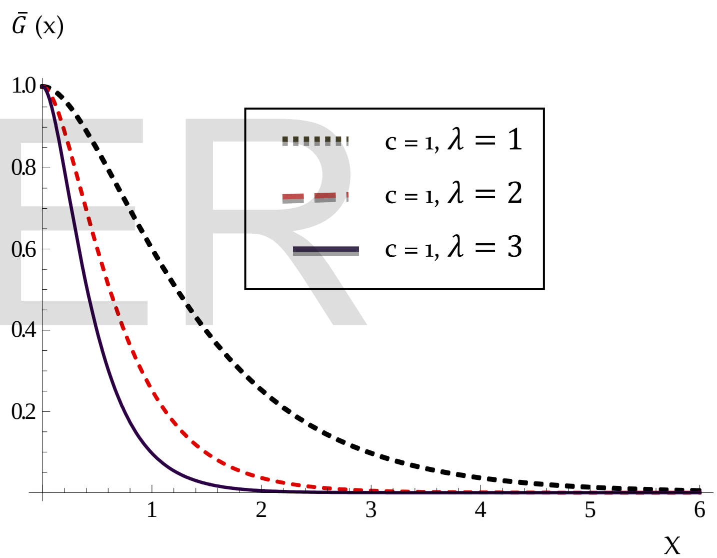

Fig. 1.4.1: The Survival Function of MMDED for the indicated values of c and λ

1.5 HAZARD RATE FUNCTION OF MMDED

The hazard function is the instant level of failure at a certain time. Characteristics of a hazard function are normally related with definite products and applications. Different hazard functions are displayed with different distribution models. The concept of this function was firstly used by Barlow (1963) and its properties were firstly investigated by Leadbetter and Watson (1964). Dhillon (1978) was another prominent name in providing consciousness about the hazard rate function. Some properties of Hazard rate were pointed out by Nadarajah and Kotz (2004). The reliability measures of weighted distributions were evaluated by Dara (2011). Hazard rate of MMDED is defined as:

$$h(x) = \frac{g(x)}{\bar{G}(x)}$$

$$h(x) = \frac{\lambda(1+c)(1-e^{-\lambda c x})e^{\lambda c x}}{(1+c)e^{\lambda c x}-1} \quad (1.5)$$

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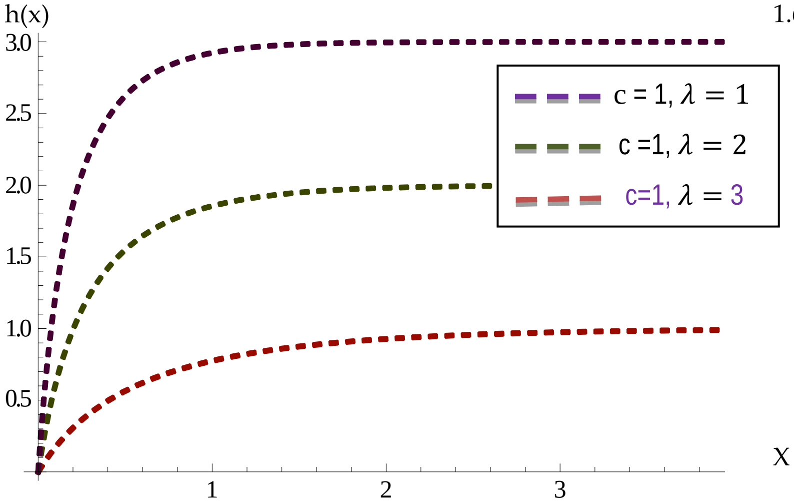

Fig. 1.5.1: The Hazard Function of MMDED for the indicated values of c and λ

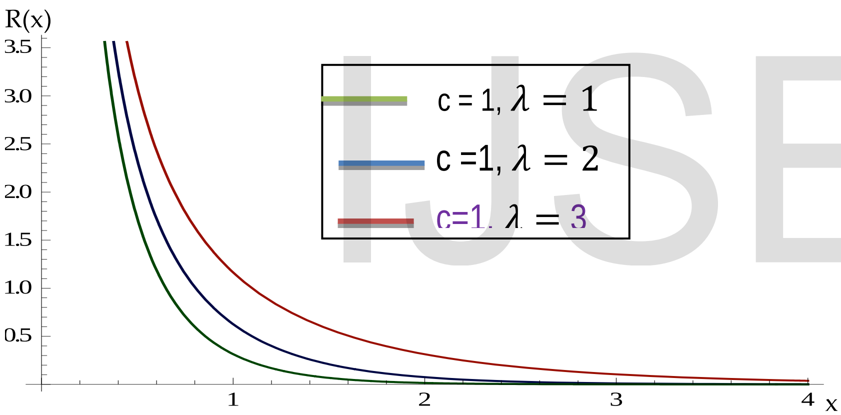

Fig. 1.6.1: Reverse Hazard Function of MMDED for the indicated values of c and λ

1.6 REVERSE HAZARD RATE FUNCTION

The reverse hazard rate function is given by

$$R(x) = \frac{g(x)}{G(x)}$$

$$R(x) = \frac{\lambda(1+c)(e^{\lambda cx}-1)}{1+ce^{\lambda cx}(e^{\lambda x}-1)-e^{\lambda cx}} \quad (1.6)$$

1.7 MILLS RATIO

The Mills Ratio is defined as:

$$m(x) = \frac{1}{h(x)}$$

$$m(x) = \frac{(1+c)e^{\lambda cx}-1}{\lambda(1+c)(1-e^{-\lambda cx})e^{\lambda cx}} \quad (1.7)$$

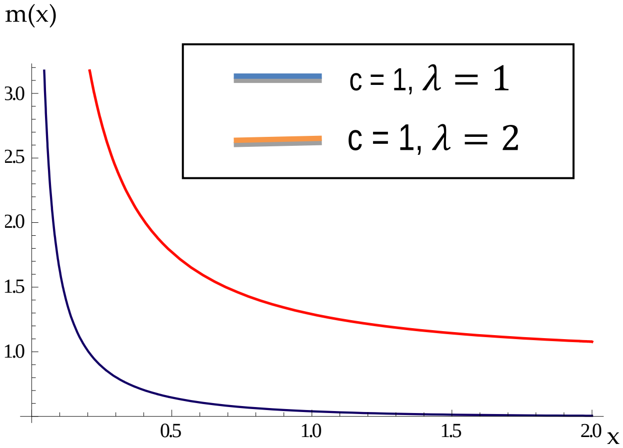

Fig. 1.7.1: Mills Ratio of MMDED for the indicated values of c and λ

1.8 MOMENT GENERATING FUNCTION OF MMDED

The moment generating function of MMDED is

$$M_X(t) = \int_0^\infty e^{tx} g(x)dx \qquad (1.8)$$

Using Eq. (1.2)

$$M_X(t) = \int_0^\infty e^{tx} . \frac{1+c}{c} \, [\lambda e^{-\lambda x} \, (1 - e^{-\lambda cx}\,]dx$$

Applying the transformations and after simplifying:

$$M_X(t) = \lambda^2 (1+c)^2 [\frac{1}{c^2(\lambda - t)} - \frac{1}{c^2(\lambda c + \lambda - t)}] \qquad (1.9)$$

1.10 CUMULANT GENERATING FUNCTION.

$$K_X\,(t) = \ln\,[M_X\,(t)] = \ln[\lambda^2(1+c)^2[\frac{1}{c^2(\lambda - t)} - \frac{1}{c^2(\lambda c + \lambda - t)}]$$

1.9 INFORMATION GENERATING FUNCTION OF MMDED

The Information Generating Function is defined as:

$$T(s) = E\,[(g(x))^{s-1}]$$

$$= \int_0^\infty (g(x))^s dx$$

Using Eq. (1.2)

$$= \frac{\lambda^s (1+c)^s}{c^s} \int_0^\infty e^{-\lambda xs} \, (1 - e^{-\lambda xc}\,)^s dx$$

Putting

$$(1 - e^{-\lambda xc}\,)^s = \sum_{i=0}^{s} (s)_i (-1)^i \, (e^{-\lambda xc}\,)^i$$

Applying the transformation and after a long simplification, the information generating function will be:

$$T(s) = \lambda^{s-1} \frac{1}{ci+s} \left(\frac{1+c}{c}\right)^s \sum_{i=0}^{s} (s)_i (-1)^i . 1 \qquad (1.10)$$

Remark 1.9.1.

For Shannon entropy $\frac{d}{ds} T(s)|_{s=0}$

1.11 LIMIT AND MODE OF MMDED

Note that the limit of the density function given in (1.2) is as follows:

$$x{\to}0\, g(x; \lambda, c) = x{\to}0 \, \frac{\lambda(1+c)}{c} \, [e^{-\lambda x} \, (1 - e^{-\lambda cx}\,)] = 0 \qquad (1.11)$$

$$x{\to}\infty\, g(x; \lambda, c) = \frac{\lambda(1+c)}{c} , x{\to}\infty \; e^{-\lambda x} \left(1 - e^{-\lambda cx}\,\right) = 0 \qquad (1.12)$$

Since

$$x{\to}\infty \; e^{-\lambda x} = 0 \text{ and } x{\to}\infty \left(1 - e^{-\lambda cx}\,\right) = 1 \qquad (1.13)$$

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1.12 MODE OF MMDED

Taking log of Eq. (1.2) on both sides:

$$\log g(x;\lambda,c) = \log\left(\frac{1+c}{c}\right) + \log\lambda - \lambda x + \log\left(1-e^{-\lambda c x}\right) \qquad (1.14)$$

Differentiating Eq. (1.14) with respect to x, we obtain:

$$\frac{\partial}{\partial x}\log g(x;\lambda,c) = -\lambda + \frac{c e^{-cx\lambda}\lambda}{1-e^{-cx\lambda}}$$

The mode of MM - Double Exponential distribution is obtained by solving the nonlinear equation with respect to x:

$$-\lambda + \frac{c e^{-cx\lambda}\lambda}{1-e^{-cx\lambda}} = 0 \qquad (1.15)$$

The mode of MMDED is given in Table 1

Table 1
Mode of MMDED

c	$\boldsymbol{\lambda}$	**mode**
2	1	1.543
2	2	0.578
2	3	0.336
2	4	0.250
2	5	0.200

Mean of MMDED

$$\mu(x) = \frac{1+c}{c\,\lambda} - \frac{1}{c\,\lambda(1+c)} \qquad (1.16)$$

Variance of MMDED

$$\sigma^2(x) = \frac{2(1+c)}{c\,\lambda^2} - \frac{2}{c\lambda^2(1+c)^2} - \frac{(1+c)^2}{c^2\lambda^2} - \frac{1}{c^2\lambda^2(1+c)^2} + 2\frac{1}{c^2\lambda^2} \qquad (1.17)$$

Table 1.2 shows the Mean, Variance and Standard Deviation (STD) with some values of the parameters λ and c.

Table 2
Mean, Variance and Standard Deviation of MMDED

c	$\boldsymbol{\lambda}$	**Mean**	**Variance**	**Standard Deviation**
1	1	1.50	1.25	1.12
1	2	0.75	0.31	0.56
1	3	0.50	0.14	0.37
2	1	1.33	1.11	1.05
2	2	0.66	0.28	0.53
2	3	0.44	0.11	0.34

1.12 MOMENTS OF MMDED

The r^th moment of MMDED is given by

$$\mu_r^{/} = E(x^r) = \frac{1+c}{c\lambda^r}\Gamma(r+1) - \frac{1}{c\lambda^r(1+c)^r}\,\Gamma(r+1)$$

for r = 1,2,3,4, the first four moments about the mean are

$\mu_1 = 0$

$$\mu_2 = \left(\frac{2(1+c)}{c\,\lambda^2} - \frac{2}{c\,\lambda^2(1+c)^2}\right) - \left(\frac{1+c}{c\lambda} - \frac{1}{c\lambda(1+c)}\right)^2$$

$\mu_3 = \left(\frac{6(1+c)}{c\,\lambda^3} - \frac{6}{c\,\lambda^3(1+c)^3}\right) - 3\left(\frac{1+c}{c\lambda} - \frac{1}{c\lambda(1+c)}\right)\left(\frac{2(1+c)}{c\,\lambda^2} -\right.$
2c λ21+c2+21+ccλ−1cλ1+c3

$\mu_4 = \left(\frac{24(1+c)}{c\,\lambda^4} - \frac{24}{c\,\lambda^4(1+c)^4}\right) - 4\left(\frac{1+c}{c\lambda} - \frac{1}{c\lambda(1+c)}\right)\left(\frac{6(1+c)}{c\,\lambda^3} -\right.$
6c λ31+c3+61+ccλ−1cλ1+c22(1+c)c λ2−2c λ21+c2−31+ccλ−1cλ1+c4

1.13 MOMENT RATIOS

Table 3
Coefficients of skewness and kurtosis of MMDED

c	λ	β_1	c	λ	β_2
1	1	2.500	1.10	1	2.90
1	2	1.140	1.15	1	2.97
1	3	0.540	1.20	1	3.04
2	2	1.110	1.21	1	3.05
2	3	0.550	1.30	1	3.18

1.14 Entropy

Entropy is considered as a major tool in every field of science and technology. In Statistics entropy is considered as an amount of incredibility. Different ideas of entropy have been given by Jaynes [55] and the entropies of continuous probability distributions have been approximated by Ma [56]. Shanon entropy is defined as H(X) of a continuous random variable X with a density function f(x)

$$H[g(x;\lambda,c)] = E[-\log g(x;\lambda,c)]$$

$$= E[-\log(1+c) - \log(c) + \log(\lambda) - \lambda x + \log(1-e-\lambda cx)$$

$$\log c - \log(1+c) - \log\lambda + \lambda E(x) - E\log(1-e^{-\lambda cx}) \quad \textbf{(1.18)}$$

using $\int_0^\infty e^{-\mu x}\log x\,dx = \frac{-1}{\mu}(c + \ln\mu)\ [Re\,\mu > 0]$

Gradshteynand Ryzhik (2007)) B1 (256)(2)

$$H(X) = E[-\log(g(X))]$$

$$= \log c - \log(1+c) - \log\lambda + \frac{c+2}{c+1} - \frac{(1+c)}{c}(c + \ln\lambda) - \frac{1}{c}[(c + \ln\lambda(1+c))]$$

1.14 MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimation of MMDED distribution may be defined as:

$$L(\lambda, c; x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} g(x_i).$$

Here the independent observations are $x_1, x_2, \dots, x_n$,then the likelihood function of the MMDED is:

$$L(\lambda, c; x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \log(g(x_i; \lambda, c))$$

$$L(\lambda, c) = n\log(1+c) - n\log(c) + n\log\lambda - \lambda\sum_{i=1}^{n} x_i + n\log(1 - e^{-\lambda c x_i}) \quad (1.19)$$

This admits the partial derivatives:

$$\frac{\partial L(\lambda,c)}{\partial\lambda} = \frac{c e^{-cx\lambda} n x}{1-e^{-cx\lambda}} + \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \quad (1.20)$$

and

$$\frac{\partial L(\lambda,c)}{\partial c} = -\frac{n}{c} + \frac{n}{1+c} + \frac{e^{-cx\lambda} n x\lambda}{1-e^{-cx\lambda}} \quad (1.21)$$

Equating these equations to zero, then we get:

$$= \frac{c e^{-cx\lambda} n x}{1-e^{-cx\lambda}} + \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \quad (1.22)$$

$$-\frac{n}{c} + \frac{n}{1+c} + \frac{e^{-cx\lambda} n x\lambda}{1-e^{-cx\lambda}} = 0 \quad (1.23)$$

which can be solved simultaneously for $\hat{\lambda}$ and $\hat{c}$.

The asymptotic variance-covariance matrix is the inverse of $\mathbf{I}(\xi, k, \theta) = -\mathbf{E}(\mathbf{H}(\mathbf{X}))$

$$\mathbf{H}(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2(\log(h(X;,\lambda,c)))}{(\partial\lambda)^2} & \frac{\partial^2(\log(h(X;\lambda,c)))}{(\partial\lambda\partial c)} \\ \frac{\partial^2(\log(h(X;\lambda,c)))}{(\partial c\partial\lambda)} & \frac{\partial^2(\log(h(X;\lambda,c)))}{(\partial c)^2} \end{pmatrix}$$

The inverse of the asymptotic covariance matrix is $\mathbf{I}(\lambda, c) = -\mathbf{E}(\mathbf{H}(\mathbf{X}))$ with

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$$\frac{\partial^2(\log(h(X;\lambda,c))}{(\partial\lambda)^2} = -\frac{c^2 e^{-2cx\lambda}nx^2}{(1-e^{-cx\lambda})^2} - \frac{c^2 e^{-cx\lambda}nx^2}{1-e^{-cx\lambda}} - \frac{n}{\lambda^2} \qquad (1.24)$$

$$\frac{\partial^2(\log(g(X;\lambda,c))}{(\partial c)^2} = \frac{n}{c^2} - \frac{n}{(1+c)^2} - \frac{e^{-2cx\lambda}nx^2\lambda^2}{(1-e^{-cx\lambda})^2} - \frac{e^{-cx\lambda}nx^2\lambda^2}{1-e^{-cx\lambda}} \qquad (1.25)$$

$$\frac{\partial^2(\log(g(X;\lambda,c))}{(\partial\lambda\partial c)} = \frac{\partial^2(\log(g(X;\lambda,c))}{(\partial c\partial\lambda)}$$

$$= \frac{e^{-cx\lambda}nx}{1-e^{-cx\lambda}} - \frac{ce^{-2cx\lambda}nx^2\lambda}{(1-e^{-cx\lambda})^2} - \frac{ce^{-cx\lambda}nx^2\lambda}{1-e^{-cx\lambda}}. \qquad (1.26)$$

1.15. Numerical Examples.

1.15.1. The Ball Bearing Data Set

See for data set published in Lawless [57].

Table 4

Ball Bearing Data Set

17.88	28.92	33.0	41.52	42.12
45.6	48.8	51.84	51.96	54.12
55.56	67.8	68.44	68.88	84.12
93.12	98.64	105.12	105.84	105.84
127.92	128.04	173.4		

Table 5

Parameters' Estimates and Goodness-of-Fit Statistics

Distributions	$\hat{\lambda}$	$\hat{c}$	A_0^2	W_0^2
Size Biased Double Weighted Exponential Distribution(SDWED)	0.028	16.647	0.833	0.134
MM-Double Exponential Distribution (MMDED)	0.0138	1.00065×10^8	2.814	0.537

In Table5, the approximations of the parameters are specified. For goodness-of-fit statistics Anderson-Darling and Cramer-von Mises tests have been used, SDWED model gives the best fitting:

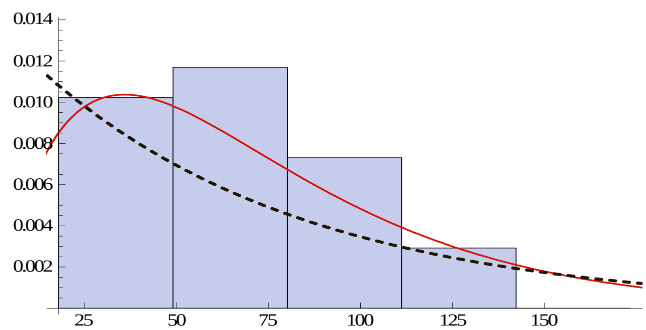

Fig 1.16.1.SDWED(solid line) and MMDED (dashed line)

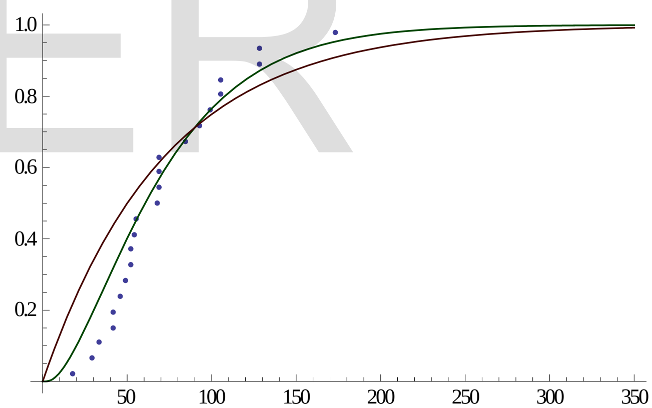

Fig.1.16.2MMDED Density Estimates, cdf Estimates and Empirical cdf

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Conclusion

In this paper,MM-Double Exponential Distribution has been introduced. The pdf of the MMDED has been studiedas well as different reliability measures such as survival function, failure rate function or hazard function. The moments, mode, the coeff.ofskewness and the coeff. of kurtosis of MMDED have been derived. For estimating the parameters of MMDED, MLE method has been used. The MMDED have been fitted to a Ball Bearing data set. In this sense we have derived the result, when we compare these two distributions SDWED and MMDED, the weighted distribution gives the best fitting and this was the object in this paper to prove that the weighted distributions are best fitted.

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